SOLOVIA REVISITED.

- An Extension of the Solowian Growth Model -

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Abstract.

In this study an Extended Version of the Solowian growth model is presented and its properties are compared with those of the original model. In the Solowian model the forces behind the growth process are exogenous and disembodied. In the Extended Model technological change is divided into two parts. One disembodied component related to the input of labour, measured in efficiency units, and one embodied component related to technical change. The disembodied labour component is endogenously determined by the real wage rate, while the embodied technical component is exogenous. The principal result of the study is that in the short and medium term a change in the savings ratio has a larger effect on the rate of growth in the Extended Model than in the original Solowian model. Also, while the equilibrium growth rate of production and that of the capital stock are the same in the original model, this is not the case in the Extended Model. In the Extended Model the equilibrium growth rate of production is smaller than that of the capital stock, measured in efficiency units.

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1. INTRODUCTION.

Technological change in the Solowian growth model (SGM) is assumed to be exogenous, labour augmenting and disembodied (Solow 1956). Furthermore, constant returns to scale and diminishing returns of each separate factor of production are assumed, i.e. the production function is assumed to be neo-classical. Technological change in the SGM is usually expressed as a residual in a linear homogenous Cobb - Douglas production function. The residual comprises everything that affects the level of production except the *volume* of labour and capital inputs.[2] A central result in the SGM is the existence of a stable long run equilibrium characterised by a shared constant rate of growth in production, consumption and the capital stock. The *levels* of these equilibria growth paths are determined by the savings ratio. The shared *rate of growth* is determined by the rate of change in the input of labour plus the rate of technological change, expressed as labour augmentation.

The golden rule model (GR) developed by Phelps (1961) is closely related to the Solowian model. GR defines a unique savings ratio yielding the highest achievable level of equilibrium growth in *consumption.*

A variety of models, where parts of the residual have been made endogenous, has since been published, i.e. *Endogenous Growth Models* (Romer 1994). Most of these models, however, rely on a seemingly unrealistic assumption of increasing returns to scale in the aggregate production function.

The aim of this paper is to present an extended version of the Solowian growth model (ESGM), retaining the assumption of constant returns to scale and still incorporating an important aspect of endogenous growth. Technological change in the ESGM is divided into two parts. One disembodied component related to the input of labour, measured in efficiency units, and one embodied component related to technical change in new capital equipment. Labour input, measured in efficiency units, is assumed to be an increasing function of the rate of real wages, while the embodied technical component is exogenously determined.

The result of the study is that the ESGM is compatible with the concept of equilibrium growth and that GR, in a modified form, can be derived in the ESGM. Numerical simulations comparing the two models show, however, that the savings ratio as an explanatory variable in the growth process changes emphasis in the ESGM. It is still the case that the *levels* of equilibrium growth are determined by the savings ratio and that the *rate of growth* in the long run is determined by the parameters of the model. In the short and medium term, however, an increase in the savings ratio has a larger effect on the rate of growth in the ESGM than in the SGM. Also, the GR savings ratio is higher in the ESGM than in the SGM. Finally, while the equilibrium growth rate of production and the capital stock are the same in the SGM, this is not the case in the ESGM. In the ESGM the equilibrium growth rate of production is *smaller* than that of the capital stock, measured in efficiency units.

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[2] A virtue of the linear homogenous Cobb - Douglas production function is that the residual can easily be statically estimated and that its content can be interpreted as either labour augmenting or capital augmenting (Phelps 1967).

2. THE SOLOWIAN GROWTH MODEL AND THE GOLDEN RULE.

In this section a standard version of the SGM, based on a linear homogenous Cobb - Douglas production function with neutral technological progress, is reviewed. Let Q(t), K(t) and N(t) denote production, capital stock and employment. Let ε denote the rate of technological progress and let α and (1- α) denote the output elasticites of capital and labour.

(1) $Q(t) = A * e^{\varepsilon t} * K(t)^{\alpha} * N(t)^{(1-\alpha)}$

Let s denote the savings ratio, C(t) consumption, δ the rate of capital depreciation and n the rate of change in employment. The change in capital stock over time (its time derivative) is denoted K'(t).

(2) $K'(t) = s * Q(t) - \delta * K(t)$

(3) $N(t) = N_0 * e^{n t}$

(4) $C(t) = (1-s) * Q(t)$

The form of the production function in equation (1) above makes it possible to treat technological progress as if it were labour augmenting. In equation (5) below we introduce a measure of labour input in efficiency units L(t). We include the rate of labour augmentation in the labour input variable. We can then reformulate the production function in equation (6) below.

(5) $L(t) = N_0 * e^{\lambda t}$ where $\lambda = n + \varepsilon / (1 - \alpha)$

(6) $Q(t) = A * K(t)^{\alpha} * L(t)^{(1-\alpha)}$

Also, introduce the variables q(t) and k(t) as defined below

(7) $q(t) = Q(t) / L(t)$ and $k(t) = K(t) / L(t)$.

We refer to k(t) as the *capital intensity* and once more reformulate the production function.

(8) $q(t) = A * k(t)^{\alpha} = f\{ k(t) \}$

Note that $f'' < 0$ because $\alpha < 1$, i.e. the function f is convex. (See the diagram below).

Now divide equation (2) by L(t) and let k'(t) denote the time derivative of k(t) = K(t) / L(t). After simplifying we get

(9) $k'(t) = s * f\{ k(t) \} - (\lambda + \delta) * k(t)$

When $k'(t) = 0$ the capital intensity $k(t)$ is constant $= k^*$.

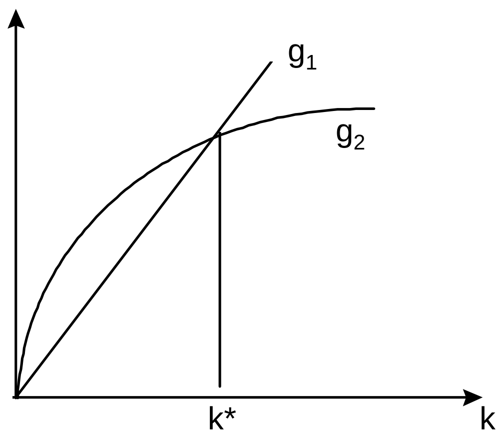
(10) $s * f(k^*) - (\lambda+\delta) * k^* = 0 \quad \Rightarrow$

(10') $k^* = \{ (\lambda+\delta) / (s * A) \}^{1/(\alpha-1)}$

k^* determines a corresponding constant value $q^* = f(k^*)$. From equation (7) above it follows that $Q(t) = q^* * L(t)$; That $C(t) = (1-s) * Q(t)$ and that $K(t) = k^* * L(t)$. Thus, when the economy is in a state where $k=k^*$ production, consumption and the capital stock will grow at a uniform rate in line with $L(t)$. That is, at the rate $\lambda = n+\varepsilon / (1-\alpha)$. This is *the equilibrium growth rate.*

Define $g_1(k) = (\lambda+\delta) * k$ and $g_2(k) = s * f(k)$. Then study equation (9) in the diagram below. k' is represented as the difference between the two curves, i.e. $k' = g_2 - g_1$. With an initial capital intensity $k(0) < k^*$ we observe that $g_2 > g_1$ which implies that $k' > 0$. As a consequence k increases. With an initial capital intensity $k(0) > k^*$ we observe that $g_2 < g_1$. This implies that $k' < 0$. Consequently, k decreases.

Thus, k represents a stable long term equilibrium towards which the economy is pushing irrespective of its initial position*.



We now turn to a numerical simulation of a *discrete version of the model* using values corresponding to the state of the Swedish economy at the present time.

(1') $Q(t) = A * (1+\varepsilon)^t * K(t)^{\alpha} * N(t)^{(1-\alpha)}$

(2') $K(t+1) = K(t) + s * Q(t) - \delta * K(t)$

(3') $N(t) = N_0 * (1+n)^t$

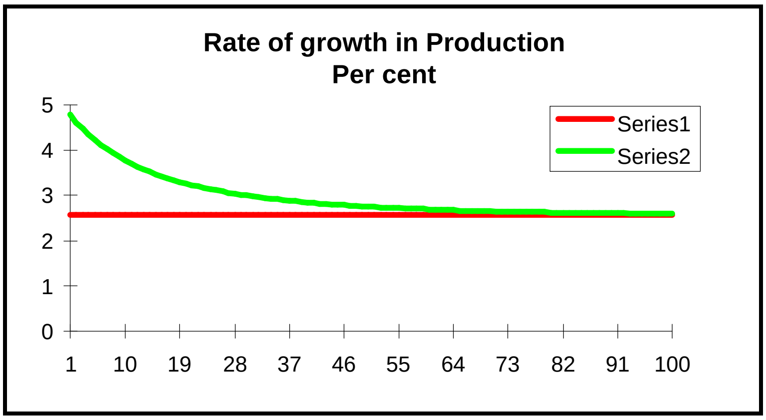
(4') $C(t) = (1-s) * Q(t)$

Initial Production Q(0)	2.09	Crowns (thousand bn)
Initial Capital stock K(0)	5.85	Crowns (thousand bn)
Initial Employment N(0)	6.95	Hours (bn)
Output elasticity of capital α	0.35	
Rate of technological change ε	0.02	
Depreciation rate δ	0.035	
Rate of change in employment n	-.005	

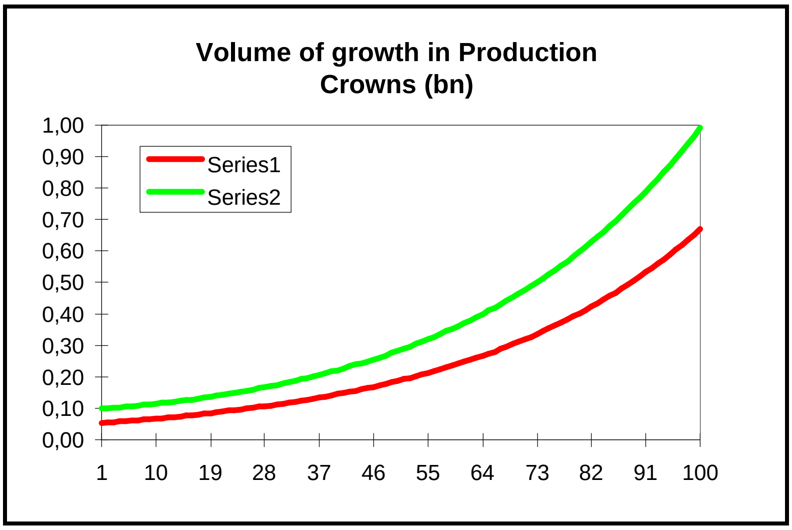
We compare two alternatives for the savings ratio: s_1 = 0.17 and s_2 = 0.35. The rate of equilibrium growth is independent of the savings ratio. It is 2.6% in both alternatives. The initial capital intensity is k(0) = 0.84.

The long run equilibrium value for the capital intensity is different in the two alternatives: k^*_1= 0.84 and k^*_2 = 2.55. Thus, in the first alternative the economy is in equilibrium growth already at the outset because k^*_1 = k(0). In the second alternative the economy is below its long run equilibrium because k^*_2 > k(0).

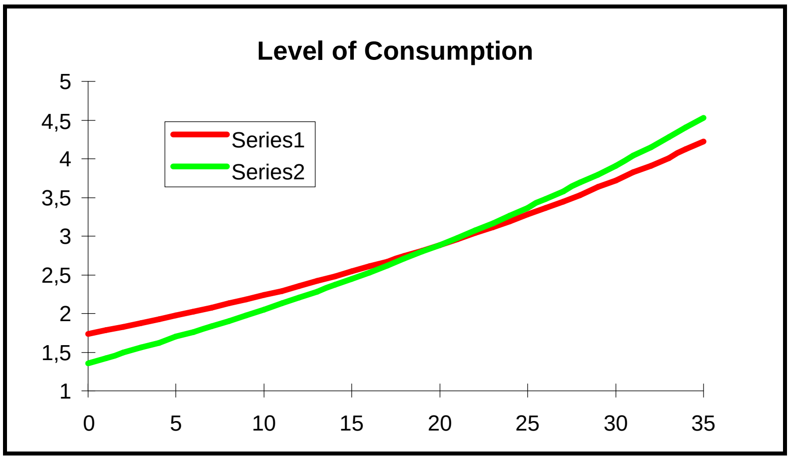
The results of the simulation for the rate of growth in production are presented in the diagram below. In the first alternative the rate of growth is of course constant at 2.6%. In the second alternative the rate of growth is initially pushed to 4.8%. This positive effect on growth, caused by the increase in the savings ratio, endures for a long time before the rate of growth subsides to its equilibrium value.



The development of the annual *volume of growth* in production is presented in the diagram below. Note, that even when the rate of growth is the same in both alternatives the annual increase in the volume of production remains higher in the second alternative. In one hundred years the rate of growth in the two alternatives has - to the last decimal - converged to the same equilibrium growth rate. The simulation, however, shows that the volume of production in the year 100 increases by 1 bn crowns in the second alternative as compared to 0.7 bn crowns in the first. The distinction between the *rate of growth* and the *volume of growth* is important to keep in mind.



The increase in the savings ratio in the second alternative reduces the initial scope for consumption compared with the first alternative. The development of the level of consumption is presented in the diagram below. It takes 20 years for the second alternative to catch up with the first, but then the second alternative, with the higher savings ratio, will forever generate a higher level of consumption than that of the first.



The fact that an upward shift in the savings ratio can increase the scope for consumption in the long run is thus demonstrated. The question is, however, if there exists an upper limit to the savings ratio, where a further increase is no longer profitable in terms of future scope for consumption in the long run.

The development of consumption in equilibrium growth is expressed in equation (11).

(11) $C(t) = (1\text{-}s) * f(k^*) * N_0 * e^{\lambda t}$

For different values of s equations (10) and (11) define a system of *parallel* long term equilibrium paths for consumption, i.e. these paths share the same growth rate λ but differ in terms of the intercept C(0). This intercept is determined by (1- s) $*$ f(k*) $*$ N_0. To find the dominant path we form a lagrangian Γ, taking into account the relationship between s and k* defined by equation (10).

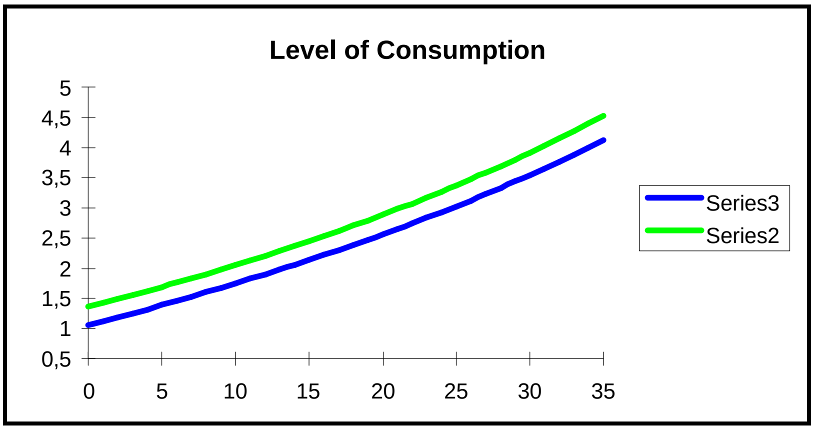
(12) $\Gamma(s,k^*,\theta) = (1- s) * f(k^*) + \theta * \{s * f(k^*) - (\lambda+\delta) * k^*\}$

Set the partial derivatives Γ_s , Γ_{k*} and Γ_θ equal to zero and simplify. We get

(13) $s = f'(k^*) * k^* / q^*$

Because $q = A * k(t)^{\alpha}$ we can conclude that the savings ratio yielding the highest equilibrium path for consumption is defined by s = α. *This is the golden rule savings ratio in the SGM.*[3]

Let us demonstrate the principle of GR in the simulation model. We compare alternative 2 with a new alternative 3, where the savings ratio is increased to 50%. Alternative 2 where the savings ratio is set at 35% is, in effect, the GR path. The result of the simulation is presented in the diagram below. The level of consumption in alternative 2 dominates alternative 3 throughout the simulation period, although alternative 3 has a *higher* savings ratio than that in alternative 2.



[3] With intertemporal optimisation the golden rule comes out as a special case, where the social rate of discount is set at zero. (Restad 1976).

3. THE EXTENDED MODEL.

In the SGM the forces behind the growth process are exogenous and disembodied. In the extended model technological change is divided into two parts: One disembodied component related to the growth of labour - measured in efficiency units - and one embodied component related to pure technical change.

The disembodied labour component is endogenously determined. Labour input - measured in efficiency units - is assumed to be an increasing function of the real wage W(t).This assumption is expressed in equation (15) below. This is a standard assumption for the development of the *volume* of the labour supply.[4] It seems, however, equally justified to assume that the *quality* of the labour supply increases with higher wages. Higher wages bring about better health and enhance possibilities for education. On the micro economic level we can refer to *efficiency wage theory* (Malcomson 1999). Higher wages induce people at work to be more efficient. Higher wages also make it easier to recruit new labour with higher skills [5]. In an economy with an open labour market there is in principle no upper bound on L(t).

The fact that endogenous growth of efficient labour is introduced might warrant the ESGM a classification as an *Endogenous Growth Model* in the Romer tradition (Romer 1994). Most models in this tradition, however, rely on an assumption of increasing returns to scale, while the ESGM does not. Furthermore, endogenous growth models treat labour efficiency in terms of a human capital *stock* while the ESGM is based on a *flow* approach.

The embodied technological component is denoted ϕ. This component is exogenously determined and is defined as the rate of improved efficiency in *new capital equipment*. This assumption is expressed in equation (16) below. Equation (14) is the production function. The ESGM retains the assumption of constant returns to scale. Finally, equation (17) below determines the real wage as the marginal productivity of labour.

(14) $Q(t) = A_0 * K(t)^{\alpha} * L(t)^{(1-\alpha)}$

(15) $L(t) = L_0 * W(t)^{\gamma} \quad ; \gamma>0$

(16) $K'(t) = s * Q(t) * e^{\phi t} - \delta * K(t)$

(17) $W(t) = Q_L(t) = A_0 * (1 - \alpha) * K(t)^{\alpha} * L(t)^{(-\alpha)}$

[4] Solow in his original paper mentions the possibility of the *volume* of employment being influenced by the wage rate. Solow, however, formulates a function for the supply of labour which leaves the properties of the model unchanged.

[5] Efficiency wage theory is often referred to in endogenous growth models, e.g. Lundborg (1999).

Use equations (15) and (17) to eliminate W(t) and express L(t) as a function of K(t). Substitute this function into (14). The result is equation (18).

(18) $Q(t) = B * K(t)^{\mu}$ where $\mu = (\alpha + \alpha\gamma) / (1+ \alpha\gamma)$.

B is a scale constant determined by initial conditions. Note that $\mu < 1$ because $\alpha < 1$.

Equation (16) can now be rewritten as

(19) $K'(t) = s * B * K(t)^{\mu} * e^{\phi t} - \delta * K(t)$

Define a function h(t) as

(20) $h(t) = K(t) / e^{\phi t / (1-\mu)}$

After some calculations equation (19) can be reformulated as

(21) $h'(t) = s * B * h(t)^{\mu} - \rho * h(t)$

where $\rho = \delta + \phi / (1-\mu)$ and $\mu = (\alpha + \alpha\gamma) / (1+ \alpha\gamma) < 1$

The fact that $\mu < 1$ means that equation (21) is a differential equation with the same properties as equation (9) above. This implies the existence of a stable stationary state h* defined by h' = 0 . *This defines equilibrium growth in the ESGM.*

(22) $h^* = \rho / (s * B)^{1/(\mu -1)}$

The definition of h(t) in equation (20) implies that the *capital stock* in equilibrium grows at the rate $\phi / (1-\mu)$. From equation (18) this in turn implies that *production* in equilibrium grows at the rate $\phi\mu / (1-\mu)$. As it is the case that $\mu < 1$ production in equilibrium grows at a *slower* rate than that of the capital stock. This effect in the ESGM is a consequence of the assumption of embodied technological progress.

Now define a function g as

(23) $g(h) = D * h^{\mu}$

Using this notation the development of consumption in equilibrium growth can be expressed as

(24) $C(t) = (1-s) * g(h^*) * e^{\phi \mu t / (1-\mu)}$

In the same way as in the SGM this equation in conjunction with equation (21) define a system of parallel long term equilibrium paths for consumption. To find the highest of these we form a lagrangian Γ, taking into account the relationship between s and h* defined by equation (21) with h' = 0.

(25) $\Gamma(s,h^*,\theta) = (1-s) * g(h^*) + \theta * \{s * g(h^*) - \rho * h^*\}$

Set the partial derivatives Γ_s , Γ_{h^*} and Γ_θ equal to zero and simplify. We get

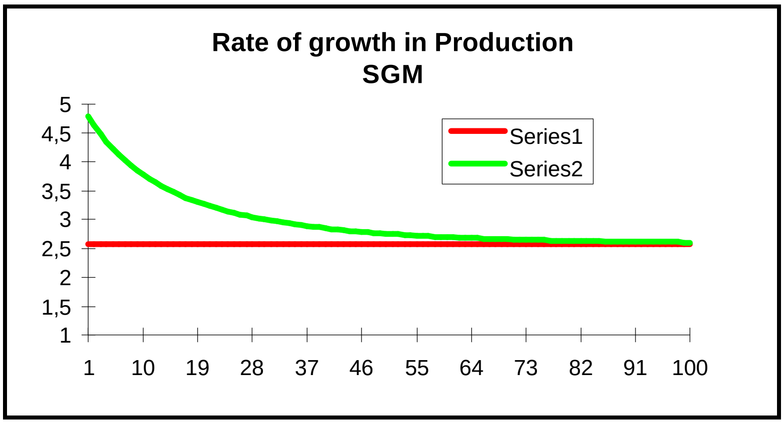
(26) $s = g'(h^*) * k^* / g(h^*)$

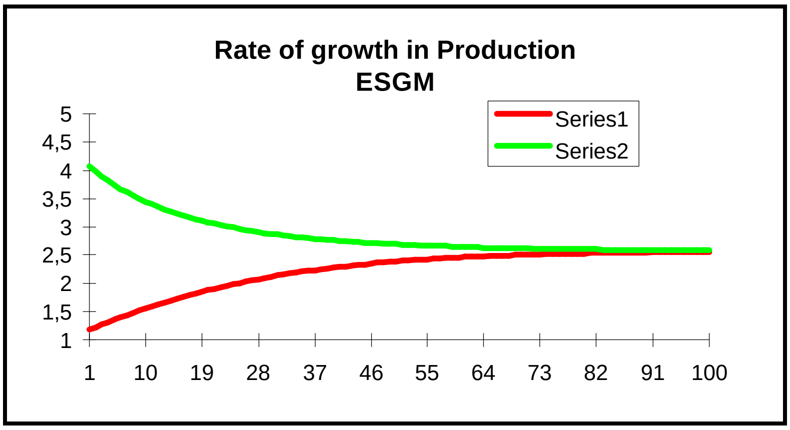
From the definition of g as $g(h) = D*h^{\mu}$ follows that $s = \mu$. *This is the GR savings ratio in the ESGM.*

As $\mu = (\alpha + \alpha\gamma) / (1+ \alpha\gamma) = \alpha (1 + \gamma) / (1+ \alpha\gamma)$ and because $\alpha < 1$ we can conclude that $\mu > \alpha$. The GR savings ratio in the SGM has been shown to be $s = \alpha$. Thus, the GR savings ratio is *larger* in the ESGM than in the SGM. This property of the ESGM is a consequence solely of the assumption of endogenous growth in effective labour ($\gamma \neq 0$) and independent of the assumption of embodied technological progress, which is also incorporated into the ESGM.

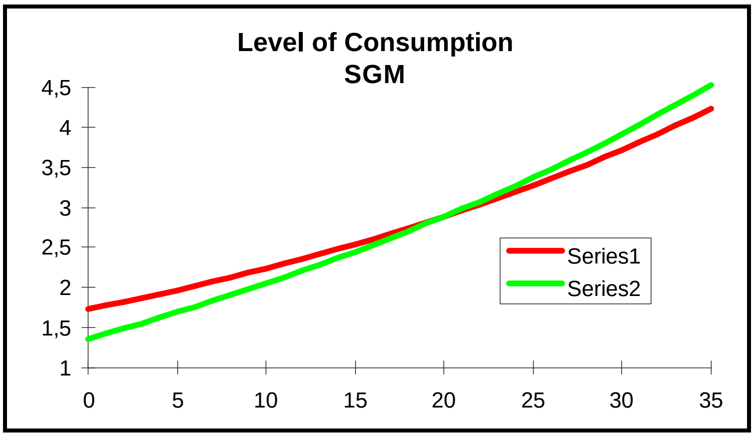
We now turn to numerical simulations and comparisons of the results for the two models. For relevant variables we use the same values as in the simulations with the SGM above. For the ESGM we add the rather arbitrary assumption that the elasticity in the L-function is $\gamma = 0.6$. We calibrate the rate of embodied technical progress ϕ so that both models generate the same rate of equilibrium growth for production, i.e. 2.6%. This is the case when the rate of efficiency growth in new capital equipment is 3%.

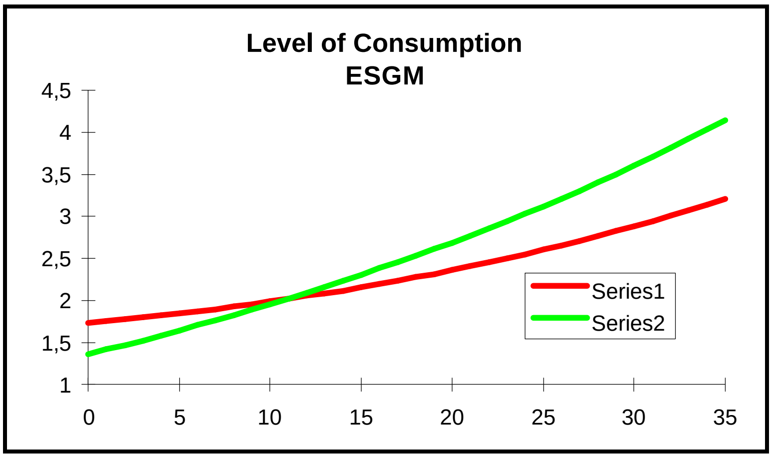
In the diagrams below we compare the development of the rate of growth in production, generated by the two models for two different savings ratios. We chose the same alternatives as above: $s_1 = 0.17$ and $s_2 = 0.35$. We start with a comparison of production growth . For the SGM we recognise the picture from above. When s = 0.17 the economy is already at the outset in a state of equilibrium growth at the rate of 2.6%. Increasing the savings ratio in the SGM to s = 0.35 boosts the initial rate of growth in production to 4.8%, i.e. by 2.2 percentage points.





In the ESGM alternative 1 implies that the economy is initially below equilibrium growth. The initial rate of growth in production is 1.1%. In alternative 2, the higher savings ratio lifts the economy above the equilibrium growth path. The initial rate of growth in production is boosted to 4%, i.e. by 2.9 percentage points. That is 0.7 percentage points *more* than in the SGM. Thus: *One and the same increase in the savings ratio has a larger effect on the rate of growth in the short and medium term in the ESGM than in the SGM.*





The effect of shifts in the savings ratio on the level of consumption in the two models is demonstrated in the two diagrams above. It is evident on inspection that one and the same increase in the savings ratio is more rewarding in terms of future levels of consumption in the ESGM than in the SGM. While the break even point is 20 years on in the SGM, it only takes 12 years for consumption to catch up in the ESGM. The difference in the level of consumption at the end of the simulation period is also much larger in the ESGM than in the SGM.

4. CONCLUSIONS AND PROBLEMS FOR FURTHER RESEARCH.

The following general conclusions can be drawn from the study:

* The ESGM generates stable equilibrium growth.

* In the ESGM, however, the equilibrium growth rate of production is smaller than that of the capital stock measured in efficiency units.

* The GR savings ratio is larger in the ESGM than in the SGM.

* One and the same increase in the savings ratio has a larger effect on the rate of growth in the short and medium term in the ESGM than in the SGM.

* One and the same increase in the savings ratio is more rewarding in terms of future levels of consumption in the ESGM than in the SGM.

The result that the ESGM generates stable equilibrium growth is of course dependent on the form of the production function, which in this paper has been specified as a linear homogenous Cobb - Douglas function. This specification makes it possible to express technological change as labour augmenting or capital augmenting as a matter of choice. See for example (Phelps 1967). In the SGM the production function can be generalised to any neo-classical production function where technological change is labour augmenting for all capital intensities. In the ESGM the production function also has to be neo-classical. Any generalised production function compatible with stable equilibrium growth in the ESGM, however, probably has to allow for technological change to be capital augmenting for all capital intensities.

Another question for further research is whether the embodiment hypothesis in the ESGM could be carried further in the vintage model tradition (Solow 1960) to also include a mechanism of endogenous capital depreciation.

Finally, the fact that the ESGM generates stable equilibrium growth probably implies that intertemporal optimisation of the social value of the stream of consumption , i.e. (Cass 1965, Restad 1976) would generate turnpike solutions similar to those based on intertemporal optimisation within the SGM.

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